Percolation games: Game Theory interpretation for limit random evolutions

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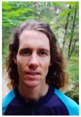

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Let's play a game

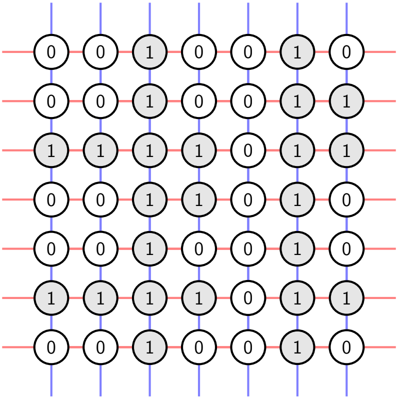


Figure 1: Average payoff game in random media

Dynamic

- State space is $\mathbb{Z}^2$
- Random reward function $G\colon \mathbb{Z}^2 \to \mathbb{R}$, where $G(z) \sim B(p)$, for $p \in [0, 1]$.
- All rewards are publicly known from the start
- Initial state is the origin $(0, 0)$
- Infinite turn-based game
- At each turn, the corresponding player chooses where to move the state:
  - Max-player chooses *up* or *down*
  - Min-player chooses *left* or *right*

Rewards and values

For the n-stage game,

$$\gamma_n(\sigma,\tau) := \frac{1}{n}\sum_{m=1}^{n} g_\omega(z_m)\,.$$

$$v_n := \max_\sigma \min_\tau \gamma_n(\sigma,\tau)\,.$$

For the ∞-stage game,

$$\gamma_\infty(\sigma,\tau) := \liminf_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} g_\omega(z_m)\,.$$

$$v_\infty := \sup_\sigma \inf_\tau \gamma_\infty(\sigma,\tau)\,.$$

Question

Does this game have a limit value?

$$(V_n) \xrightarrow[n\to\infty]{?} v_\infty \,.$$

Is v_∞ a constant?

Model flexibility

Non-essential modelling choices

- Turn-based or concurrent
- I.I.D. random environment
- Actions of players

Essential choices

- Transitions are state and time independent
- State is a group

Question

What random stochastic games in infinite spaces have a limit value?

Motivation from Analysis

Continuous environment

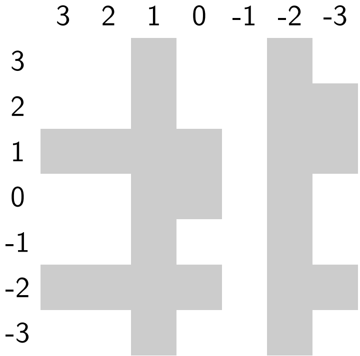


Figure 2: Continuous random environment

Differential game

- Environment are random blocks of value 0 and 1
- Dynamic

$$\begin{cases} \dot{x}_1(t) = \tau(t) \in [0,1] \\ \dot{x}_2(t) = \sigma(t) \in [0,1] \\ x(0) = x_0 = (0,0) \end{cases}$$

- Payoff

$$\int_0^T g_\omega(x(s))ds$$

- Value $U(T,0)$ is the (random) aggregation of rewards the max-player can get in T units of time

Game-theoretical question

Question

Does this game have a limit value?

$$\left(\frac{1}{T} U(T, 0) \right) \xrightarrow[T \to \infty]{?} u\,.$$

Is u a constant?

Hamilton-Jacobi equations

The value function u satisfies

$$\begin{cases} \partial_t u(t,x) - g_\omega(x) - |\partial_y u(t,x)| + |\partial_x u(t,x)| = 0 \\ u(0,x) = u_0(x) \end{cases}$$

which can be written as

$$\begin{cases} \partial_t u(t,x) + H_\omega(\nabla_x u(t,x), x) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Consider the space-accelerated equation

$$\begin{cases} \partial_t u^{(T)}(t,x) + H_\omega(\nabla_x u^{(T)}(t,x), Tx) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Analysis question

Question

Does this PDE have a limit solution?

$$\left(\frac{1}{T} U^{(T)}(Tt, x) \right) \xrightarrow[n \to \infty]{?} u(t, x) \,.$$

What is the limit PDE?

$$\begin{cases} \partial_t u(t, x) + \overline{H}(\nabla_x u(t, x)) = 0 \\ u(0, x) = u_0(x) \end{cases}$$

Going back to games

Consider linear initial conditions u_0. Then,

$$U^{(T)}(t, x) := \frac{1}{T} U^{(1)} \left(Tt, Tx \right) .$$

In particular, if H homogenizes,

$$u(1, 0) = \lim_{T \to \infty} \frac{1}{T} U^{(1)} \left(T, 0 \right) .$$

Necessary condition for Analysis limit

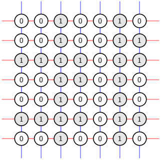
For H_ω to have a limit, it is required that
the random differential game has a limit value.

Question

What random differential games have a limit value?

Back to our game

Let's play a game



Question

Does this game have a limit value?

$$(V_n) \xrightarrow[n\to\infty]{?} v_\infty \,.$$

Is v_∞ a constant?

Critical thresholds

Theorem (Critical thresholds)

There exists $0 < p_0 < p_1 < 1$ *such that*

$$(V_n) \xrightarrow[n \to \infty]{} 0 \qquad \forall p < p_0$$

$$(V_n) \xrightarrow[n \to \infty]{} 1 \qquad \forall p > p_1$$

Translation to directed percolation

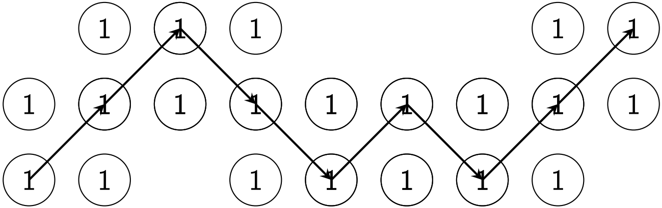


Figure 3: Structure that guarantees a value of one for the max-player

Percolation model

In the oriented percolation model,

- Each node may have two edges (northeast and southeast).
- Each edge may appear independently with probability p.

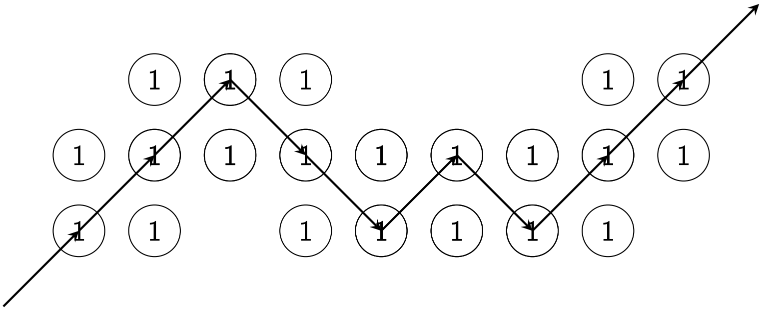
Theorem (Critical percolation parameter)

There exists $0.6298 \leq p_c \leq 2/3$ *such that, in the percolation model with parameter* $p > p_c$*, the probability that there is an infinite path starting at the origin is strictly positive.*

Technical extension

To prove our result on games, we must

- Deduce the existence of an infinite line somewhere in the grid.
- This infinite line is more or less horizontal.



Adding time

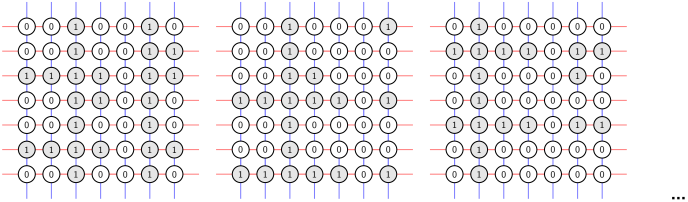


Figure 4: Game on $\mathbb{Z}^3$, advancing in the time axis

Time introduces independence over time!

Convergence in probability

Theorem

For all $p \in [0, 1]$, there exists a constant limit value. Formally,

$$(V_n) \xrightarrow[n\to\infty]{\mathbb{P}} v_\infty \in \mathbb{R} \,.$$

Proof steps

- V_n concentrates on its expectation $\mathbb{E}(V_n)$
- $(\mathbb{E}(V_n)) \xrightarrow[n\to\infty]{} v_\infty$
- $(\mathbb{E}(V_n))_{n\in\mathbb{N}}$ converge fast to v_∞
- Therefore, V_n concentrates on v_∞

The proof technique does not generalize if there is a lot of dependence on the past.

Why transient?

We rely on Azuma's inequality.

Lemma (Concentration of martingales)

Let $(X_n)_{n\in\mathbb{N}}$ be a martingale and $(c_n)_{n\in\mathbb{N}}$ a real sequence such that, for all $n \in \mathbb{N}$, $|X_n - X_{n+1}| \leq c_n$ almost surely. Then, for all $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\mathbb{P}(|X_n - X_0| \geq \varepsilon) \leq 2\exp\left(\frac{-\varepsilon^2}{2\sum_{m=0}^{n-1} c_m^2}\right) .$$

Proof: Concentration on $\mathbb{E}(V_n)$

We aim to show that $\mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon)$ decreases with n. We will do so by defining a martingale and applying Azuma's inequality.

Proof: Concentration on $\mathbb{E}(V_n)$ (2)

For $m \in \mathbb{N}$,

- Define the σ-algebra

$$\mathcal{C}_m := \sigma(\{G(z,i,j) : z \in Z_m, i \in I, j \in J\})\,.$$

- Note the inequality

$$|\mathbb{E}(V_n(0)|\mathcal{C}_m) - \mathbb{E}(V_n(0)|\mathcal{C}_{m+1})| \leq \begin{cases} \frac{1}{n} & m < n \\ 0 & m \geq n \end{cases}\,.$$

- Define the martingale

$$X_m := \mathbb{E}(V_n(0)|\mathcal{C}_m)\,.$$

Proof: Concentration on $\mathbb{E}(V_n)$ (3)

Then, applying Azuma's inequality,

$$\begin{aligned}\mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon) &= \mathbb{P}(|X_n - X_0| \geq \varepsilon) \\ &\leq 2\exp\left(\frac{-\varepsilon^2}{2\sum_{m=0}^{n-1}(1/n)^2}\right) \\ &\leq 2\exp\left(\frac{-\varepsilon^2}{2}n\right).\end{aligned}$$

Therefore, V_n concentrates on $\mathbb{E}(V_n)$.

Proof steps

- V_n concentrates on its expectation $\mathbb{E}(V_n)$
- $(\mathbb{E}(V_n)) \xrightarrow[n\to\infty]{} v_\infty$
- $(\mathbb{E}(V_n))_{n\in\mathbb{N}}$ converge fast to v_∞
- Therefore, V_n concentrates on v_∞

Proof: Convergence of $\mathbb{E}(V_n)$

We aim to show that $\mathbb{E}(V_n)$ converges.
We will study the subadditivity of $(n\mathbb{E}(V_n))_{n\geq 1}$.

Lemma (Convergence of subadditive sequences)

Let $\phi\colon \mathbb{N} \to (0,\infty)$ *be an increasing function such that* $\sum_{n=1}^{\infty} \phi(n)/n^2 < \infty$, *and* $(f(n))_{n\in\mathbb{N}}$ *be a sequence such that, for all* $n \in \mathbb{N}$ *and all* $m \in [n/2, 2n]$,

$$f(n+m) \leq f(n) + f(m) + \phi(n+m)\,.$$

Then, there exists $L \in \mathbb{R}$ *such that*

$$\left(\frac{f(n)}{n}\right) \xrightarrow[n\to\infty]{} L\,.$$

Proof: Convergence of $\mathbb{E}(V_n)$ (2)

$$
\begin{aligned}
&\mathbb{P}(\exists z \in B_\infty(0,2n) \quad |V_n(z) - \mathbb{E}(V_n)| \geq \varepsilon) \\
&\quad \leq \sum_{z \in B_\infty(0,2n)} \mathbb{P}(|V_n(z) - \mathbb{E}(V_n)| \geq \varepsilon) && \text{(union sum)} \\
&\quad = \sum_{z \in B_\infty(0,2n)} \mathbb{P}(|V_n(0) - \mathbb{E}(V_n)| \geq \varepsilon) && \text{(space-homogeneity)} \\
&\quad \leq |B_\infty(0,2n)| 2 \exp\left(\frac{-\varepsilon^2}{2} n\right) && \text{(Azuma's inequality)} \\
&\quad \leq (4n+1)^3 2 \exp\left(\frac{-\varepsilon^2}{2} n\right) && \text{(Azuma's inequality)} \\
&\quad =: \psi(n, \varepsilon) .
\end{aligned}
$$

Proof: Convergence of $\mathbb{E}(V_n)$ (3)

$$\begin{aligned}
&\mathbb{E}\left(\min_{z\in B_\infty(0,2n)} V_n(z)\right) \\
&\quad \geq 0 \cdot \mathbb{P}\left(\min_{z\in B_\infty(0,2n)} V_n(z) \leq \mathbb{E}(V_n) - \varepsilon_n\right) \\
&\qquad + (\mathbb{E}(V_n) - \varepsilon_n) \cdot \mathbb{P}\left(\min_{z\in Z^{(2n)}} V_n(z) \geq \mathbb{E}(V_n) - \varepsilon_n\right) \\
&\quad \geq (1 - \psi(n, \varepsilon_n)) \cdot \mathbb{E}(V_n) - \varepsilon_n \\
&\quad \geq \mathbb{E}(V_n) - (\psi(n, \varepsilon_n) + \varepsilon_n) \,.
\end{aligned}$$

Now we can show that $n\mathbb{E}(V_n)$ is subadditive enough.

Proof: Convergence of $\mathbb{E}(V_n)$ (4)

By playing by blocks, we obtain, for $m \leq 2n$,

$$
\begin{aligned}
(m+n)\mathbb{E}(V_{m+n}) &\geq m\mathbb{E}(V_m) + n\mathbb{E}\left(\min_{z \in Z^{(2n)}} V_n(z)\right) \\
&\geq m\mathbb{E}(V_m) + n\mathbb{E}(V_n) - n(\psi(n, \varepsilon_n) + \varepsilon_n) \,.
\end{aligned}
$$

which is sufficient subadditivity taking an appropiate sequence $(\varepsilon_n) \xrightarrow[n \to \infty]{} 0$.
Therefore, there exists v_∞ such that

$$
\mathbb{E}(V_n) \xrightarrow[n \to \infty]{} v_\infty \,.
$$

Proof steps

- V_n concentrates on its expectation $\mathbb{E}(V_n)$
- $(\mathbb{E}(V_n)) \xrightarrow[n\to\infty]{} v_\infty$
- $(\mathbb{E}(V_n))_{n\in\mathbb{N}}$ converge fast to v_∞
- Therefore, V_n concentrates on v_∞

Proof: Fast convergence of $\mathbb{E}(V_n)$

Recall that

$$\mathbb{E}(V_{2n}) \geq \mathbb{E}(V_n) - (\psi(n, \varepsilon_n) + \varepsilon_n) \,.$$

Moreover, we can choose $\delta > 0$ such that

$$(\psi(n, \varepsilon_n) + \varepsilon_n) \in O(n^{-\delta}) \,.$$

Proof: Fast convergence of $\mathbb{E}(V_n)$

By the telescopic sum, we get for $\ell > 0$

$$\begin{aligned}
\mathbb{E}(V_{2^\ell n}) &\geq \mathbb{E}(V_n) - \sum_{\ell'=0}^{\ell-1} \mathbb{E}\left(V_{2^{\ell'} n}\right) - \mathbb{E}\left(V_{2^{\ell'+1} n}\right) \\
&\geq \mathbb{E}(V_n) - \sum_{\ell'=0}^{\ell-1} K(2^{\ell'} n)^{-\delta} \\
&\geq \mathbb{E}(V_n) - n^{-\delta} \frac{K}{1 - 2^{-\delta}} \geq \mathbb{E}(V_n) + O(n^{-\delta}) \,.
\end{aligned}$$

Therefore,

$$|v_\infty - \mathbb{E}(V_n)| \in O(n^{-\delta}) \,.$$

Proof steps

- V_n concentrates on its expectation $\mathbb{E}(V_n)$
- $(\mathbb{E}(V_n)) \xrightarrow[n \to \infty]{} v_\infty$
- $(\mathbb{E}(V_n))_{n \in \mathbb{N}}$ converge fast to v_∞
- Therefore, V_n concentrates on v_∞

Proof: Concentration on v_∞

Recall that

- $|v_\infty - \mathbb{E}(V_n)| \in O(n^{-\delta})$
- $\mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon) \leq \exp\left(\frac{-\varepsilon^2}{2} n\right)$

Therefore, there exists $K > 0$ such that

$$\begin{aligned}
&\mathbb{P}(|V_n - v_\infty| \geq \varepsilon + Kn^{-\delta}) \\
&\quad \leq \mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon + Kn^{-\delta} - |\mathbb{E}(V_n) - v_\infty|) \\
&\quad \leq \mathbb{P}(|V_n - \mathbb{E}(V_n)| \geq \varepsilon) \\
&\quad \leq \exp\left(\frac{-\varepsilon^2}{2} n\right) .
\end{aligned}$$

Convergence in probability

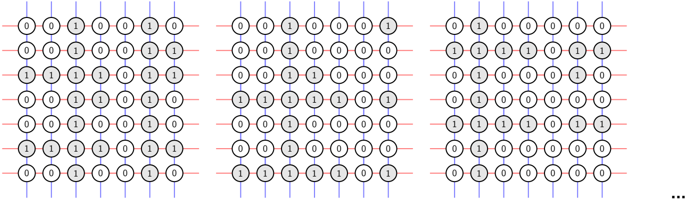


Figure 5: Game on $\mathbb{Z}^3$, advancing in the time axis

Theorem

For all $p \in [0,1]$, there exists a constant limit value. Formally,

$$(V_n) \xrightarrow[n\to\infty]{\mathbb{P}} v_\infty \in \mathbb{R}\,.$$

Extension: Forced plays

Let $\varepsilon > 0$. Define the set

$$Z_m \approx \{z \in \mathbb{Z}^2 : ||z||_2 \leq m^{(1+\varepsilon)1/2} - 1\} .$$

Restrict the players from entering Z_m at stage m. Then, there exists $K, \delta > 0$ such that for all $\varepsilon > 0$

$$\mathbb{P}(|V_n - v_\infty| \geq \varepsilon + Kn^{-\delta}) \xrightarrow[n\to\infty]{} 0 .$$

Let's play a game

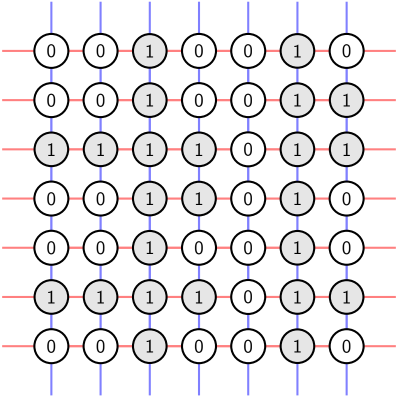


Figure 6: Average payoff game in random media

References I